Journal of Engineering Physics and Thermophysics, Vol. 79, No. 3, 2006

LINEARIZATION FOR A NONLINEAR HEAT-CONDUCTION EQUATION

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UDC 531.2

A method of linearizing a quasilinear heat-conduction equation is proposed. The closeness of the exact and approximate solutions was estimated by an inequality, the right side of which determines the closeness of the linearized solution to the initial condition.

We will consider the following quasilinear heat-conduction equation:

$$\frac{\partial \psi}{\partial t} = a^2 \Delta \psi + a^2 F\left(x,\, \psi,\, \frac{\partial \psi}{\partial x}\right), \tag{1}$$

where $x = (x_1,\, x_2,\, x_3)$. Let us write the following initial and boundary conditions for (1):

$$\psi\,(x,0) = f\,(x)\,, \quad x \in D\,, \quad \psi\,(x,t)\Big|_{x \in S = \partial D} = g\,(x,t)\,. \tag{2}$$

It will be assumed that the function $F(x,\, \psi,\, \partial\psi/\partial x)$ is continuously differentiated with respect to its arguments. In this case, problem (1)–(2) can be solved approximately with the use of the linearization

$$\frac{\partial \overline{\psi}}{\partial t} = a^2 \Delta \overline{\psi} + a^2 \overline{F}\left(x,\, \overline{\psi},\, \frac{\partial \overline{\psi}}{\partial x}\right), \tag{3}$$

$$\overline{\psi}\,(x,0) = f\,(x)\,, \quad \overline{\psi}\,(x,t)\Big|_{x \in S = \partial D} = g\,(x,t)\,. \tag{4}$$

Here

$$\overline{F}\left(x,\, \psi,\, \frac{\partial \psi}{\partial x}\right) = F\left(x,\, \psi\,(x,0)\,,\, \frac{\partial \psi\,(x,0)}{\partial x}\right) + \frac{\partial F\left(x,\, \psi\,(x,0)\,,\, \frac{\partial \psi\,(x,0)}{\partial x}\right)}{\partial \psi\,(x,0)}\,(\psi - \psi\,(x,0)) +$$

$$+ \sum_{i=1}^{3} \frac{\partial F\left(x,\, \psi\,(x,0)\,,\, \frac{\partial \psi\,(x,0)}{\partial x}\right)}{\partial \psi'_{x_i}\,(x,0)} \left(\frac{\partial \psi}{\partial x_i} - \frac{\partial \psi\,(x,0)}{\partial x_i}\right) \tag{5}$$

or

$$\overline{F}\left(x,\, \psi,\, \frac{\partial \psi}{\partial x}\right) = F\left(x, f\,(x)\,,\, \frac{\partial f\,(x)}{\partial x}\right) + \frac{\partial F\left(x, f\,(x)\,,\, \frac{\partial f\,(x)}{\partial x}\right)}{\partial f\,(x)}\,(\psi - f\,(x)) +$$

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$$+\sum_{i=1}^{3}\frac{\partial F\left(x, f(x), \frac{\partial f(x)}{\partial x}\right)}{\partial f'_{x_i}(x)}\left(\frac{\partial \psi}{\partial x_i}-\frac{\partial f(x)}{\partial x_i}\right). \tag{6}$$

Let us estimate the accuracy of the exact and approximate solutions of problems (1)–(2) and (3), (4), and (6). For this purpose, we will denote the difference between them by $v(x, t)$:

$$v(x, t)=\psi(x, t)-\overline{\psi}(x, t). \tag{7}$$

Then we will write the following problem for $v(x, t)$:

$$\frac{\partial v}{\partial t}=a^2\Delta v+a^2\left(F\left(x, \psi, \frac{\partial \psi}{\partial x}\right)-F\left(x, f(x), \frac{\partial f}{\partial x}\right)-\frac{\partial F\left(x, f(x), \frac{\partial f(x)}{\partial x}\right)}{\partial f_i(x)}\times\right.$$

$$\left.\times\left(\overline{\psi}-f(x)\right)-\sum_{i=1}^{3}\frac{\partial F\left(x, f(x), \frac{\partial f(x)}{\partial x}\right)}{\partial x_i}\left(\frac{\partial \overline{\psi}}{\partial x_i}-\frac{\partial f(x)}{\partial x_i}\right)\right), \tag{8}$$

$$v(x, 0)=0, \quad v(x, t)\big|_{x\in S=\partial D}=0. \tag{9}$$

Let us multiply both sides of Eq. (8) by $v(x, t)$ and integrate them over the region D. As a result, we obtain

$$\iiint_D v(x, t)\left[v'_t(x, t)-a^2\Delta v(x, t)\right]dx=\iiint_D v(x, t)\left[a^2\left(F\left(x, \psi, \frac{\partial \psi}{\partial x}\right)-\overline{F}\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right)\right)\right]dx; \tag{10}$$

let us introduce the designations

$$\|v\|^2\overset{df}{=}\iiint_D v^2(x, t)\,dx, \tag{11}$$

$$\|v\|_\Delta^2\overset{df}{=}\iiint_D\sum_{i=1}^{3}(\partial_i v(x, t))^2\,dx. \tag{12}$$

It follows from formula (6) that

$$\left|F\left(x, \psi, \frac{\partial \psi}{\partial x}\right)-\overline{F}\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right)\right|=\left[F\left(x, \psi, \frac{\partial \psi}{\partial x}\right)-F\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right)\right]+\left[F\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right)-F\left(x, f, \frac{\partial f}{\partial x}\right)\right]-$$

$$-\frac{\partial F\left(x, f, \frac{\partial f}{\partial x}\right)}{\partial f}(\overline{\psi}-f)-\sum_{i=1}^{3}\frac{\partial F\left(x, f, \frac{\partial f}{\partial x}\right)}{\partial f'_{x_i}}\left(\frac{\partial \overline{\psi}}{\partial x_i}-\frac{\partial f}{\partial x_i}\right). \tag{13}$$

We will assume that the function $F(x, \psi, \partial\psi/\partial x)$ is continuously differentiated and satisfies the Lipschitz condition:

$$\left| F\left(x, \psi, \frac{\partial \psi}{\partial x}\right) - F\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right) \right| \le L \left[|\psi - \overline{\psi}| + \sum_{i=1}^{3} \left| \frac{\partial \psi}{\partial x_i} - \frac{\partial \overline{\psi}}{\partial x_i} \right| \right] = L \left[|v| + \sum_{i=1}^{3} \left| \frac{\partial v}{\partial x_i} \right| \right],$$

$$\left| F\left(x, \overline{\psi}, \frac{\partial \overline{\psi}}{\partial x}\right) - F\left(x, f, \frac{\partial f}{\partial x}\right) \right| \le L \left[|\overline{\psi} - f| + \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right| \right], \tag{14}$$

$$\left| \frac{\partial F\left(f, \frac{\partial f}{\partial x}\right)}{\partial f} (\overline{\psi} - f) \right| \le M\, |\overline{\psi} - f|\,, \quad \left| \sum_{i=1}^{3} \frac{\partial F\left(f, \frac{\partial f}{\partial x}\right)}{\partial f'_{x_i}} \left(\frac{\partial \overline{\psi}}{\partial x_i} - \frac{\partial f}{\partial x_i} \right) \right| \le M \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right|.$$

Let us substitute formulas (11), (12), and (14) into formula (10):

$$\frac{1}{2} \frac{d}{dt} \iiint_D v^2 (x, t)\, dx + a^2 \iiint_D \sum_{i=1}^{3} \left[\left(\frac{\partial v}{\partial x_i} \right)^2 - \frac{\partial}{\partial x_i} \left(v \frac{\partial v}{\partial x_i} \right) \right] dx \equiv \frac{1}{2} \frac{d}{dt} \|v\|^2 + a^2 \|v\|_\Delta^2 \le$$

$$\le \iiint_D |v(x, t)| \left(L \left[|v| + \sum_{i=1}^{3} \left| \frac{\partial v}{\partial x_i} \right| \right] + L \left[|\overline{\psi} - f| + \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right| \right] + M\, |\overline{\psi} - f| + M \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right| \right) dx\,, \tag{15}$$

$$M = \max_{x \in D} \left[\left| \frac{\partial F\left(x, f, \frac{\partial f}{\partial x}\right)}{\partial f} \right|, \left| \frac{\partial F\left(x, f, \frac{\partial f}{\partial x}\right)}{\partial f'_{x_i}} \right| \right], \quad i = 1, 2, 3\,. \tag{16}$$

Moreover,

$$\iiint_D |v(x, t)|\, L \left[|v| + \sum_{i=1}^{3} \left| \frac{\partial v}{\partial x_i} \right| \right] dx = L \iiint_D \left((v)^2 + \sum \frac{\partial}{\partial x_i} \frac{v^2}{2} \right) dx = L\, \|v\|^2\,, \tag{17}$$

since $v = 0$ in S and $\iiint_D \frac{\partial}{\partial x_i} \frac{v^2}{2} dx = \iint_D \frac{v^2}{2} \cos\,(n, x_i)\, ds = 0$. Let us now use the relation

$$|ab| \le \frac{\alpha}{2\beta} a^2 + \frac{\beta}{2\alpha} b^2\,, \tag{18}$$

where α and β are positive constants differing from zero. Therefore,

$$\iiint_D |v(x, t)| \left(L \left(|\overline{\psi} - f| + \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right| \right) + M\, |\overline{\psi} - f| + M \sum_{i=1}^{3} \left| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right| \right) dx \le$$

$$\le \frac{\alpha}{2\beta} L\, \|v\|^2 + \frac{\beta}{2\alpha} L\, \|\overline{\psi} - f\|^2 + L\, \|\overline{\psi} - f\|_\Delta^2 + M \left(\frac{\alpha}{2\beta} \|v\|^2 + \frac{\beta}{2\alpha} \|\overline{\psi} - f\| \right) +$$

$$+ M \left(\frac{\alpha}{2\beta} \|v\|^2 + \sum_{i=1}^{3} \frac{\beta}{2\alpha} \left\| \frac{\partial}{\partial x_i} (\overline{\psi} - f) \right\|^2 \right). \tag{19}$$

The following inequality follows from the above-indicated formulas:

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + a^2\|v\|_\Delta^2 \le \frac{\alpha}{2\beta}L\|v\|^2 + \frac{\beta}{2\alpha}L\|\overline{\psi}-f\|^2 + L\|\overline{\psi}-f\|_\Delta^2 +$$

$$+M\left(\frac{\alpha}{2\beta}\|v\|^2 + \frac{\beta}{2\alpha}L\|\overline{\psi}-f\|\right) + M\left(\frac{\alpha}{2\beta}\|v\|^2 + \sum_{i=1}^{3}\frac{\beta}{2\alpha}\left\|\frac{\partial}{\partial x_i}(\overline{\psi}-f)\right\|^2\right) \tag{20}$$

or

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + C\|v\|^2 \le h(t)\,, \tag{21}$$

where

$$C = a^2 - M\frac{\alpha}{\beta} - L\frac{\alpha}{2\beta}\,; \tag{22}$$

$$h(t) = \frac{\beta}{2\alpha}L\|\overline{\psi}-f\|^2 + L\|\overline{\psi}-f\|_\Delta^2 + M\left(\frac{\beta}{2\alpha}\|\overline{\psi}-f\|^2 + \sum_{i=1}^{3}\frac{\beta}{2\alpha}\left\|\frac{\partial}{\partial x_i}(\overline{\psi}-f)\right\|^2\right). \tag{23}$$

Formula (21) represents the known Bellman–Gronwall inequality [2]. We will rewrite it in the following form:

$$\|v\|^2 \le A\int_0^t \|v\|^2\,d\tau + g(t)\,, \tag{24}$$

where

$$A = (2M+L)\frac{\alpha}{\beta} - 2a^2\,;\quad g(t) = 2h(t) = \left(\frac{\beta}{\alpha}(L+M) + 2L\right)\|\overline{\psi}-f\|^2 + M\sum_{i=1}^{3}\frac{\beta}{\alpha}\left\|\frac{\partial}{\partial x_i}(\overline{\psi}-f)\right\|^2. \tag{25}$$

Here, the constants α and β are selected such that the condition $A > 0$ is fulfilled. In this case, inequality (24) can be solved by the successive-substitution method, which makes it possible to obtain the following inequality for estimating the closeness of the exact and approximate solutions of the initial problem:

$$\|v\|^2 \le 2h(t) + 2A\int_0^t h(\tau)\exp(A(t-\tau))\,d\tau\,, \tag{26}$$

where $h(\tau)$ is determined from formula (24).

A solution of the initial problem (1)–(2) can be verified by the method of successive approximations.

NOTATION

a^2, thermal diffusivity; L, Lipschitz constant of the function F; F, amount of heat generated (lost) in a unit volume for a unit time (entropy); ψ, thermal response of a medium (change in the temperature).

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